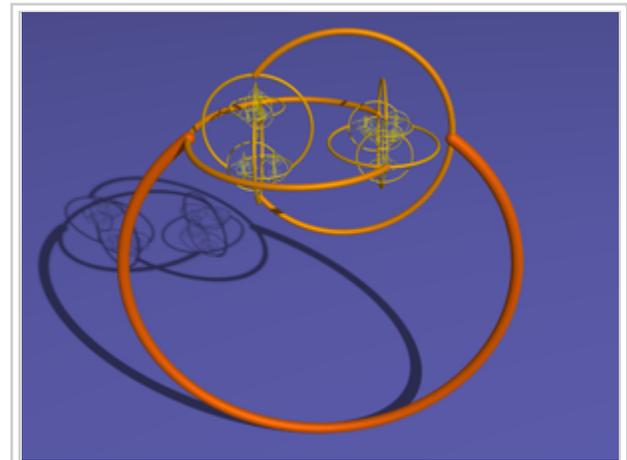# Alexander horned sphere

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The **Alexander horned sphere** is a pathological object in topology discovered by J. W. Alexander (1924).

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Alexander horned sphere

Construction

The Alexander horned sphere is the particular embedding of a sphere in 3-dimensional Euclidean space obtained by the following construction, starting with a standard torus:[1]

1. Remove a radial slice of the torus.
2. Connect a standard punctured torus to each side of the cut, interlinked with the torus on the other side.
3. Repeat steps 1–2 on the two tori just added *ad infinitum*.

By considering only the points of the tori that are not removed at some stage, an embedding results of the sphere with a Cantor set removed. This embedding extends to the whole sphere, since points approaching two different points of the Cantor set will be at least a fixed distance apart in the construction.

Impact on theory

The horned sphere, together with its inside, is a topological 3-ball, the **Alexander horned ball**, and so is simply connected; i.e., every loop can be shrunk to a point while staying inside. The exterior is *not* simply connected, unlike the exterior of the usual round sphere; a loop linking a torus in the above construction cannot be shrunk to a point without touching the horned sphere. This shows that the Jordan–Schönflies theorem does not hold in three dimensions, as Alexander had originally thought. Alexander also proved that the theorem does hold in three dimensions for piecewise linear/smooth embeddings. This is one of the earliest examples where the need for distinction between the topological

category of manifolds, and the categories of differentiable manifolds, and piecewise linear manifolds was noticed.

Now consider Alexander's horned sphere as an embedding into the 3-sphere, considered as the one-point compactification of the 3-dimensional Euclidean space $\mathbf{R}^3$. The closure of the non-simply connected domain is called the **solid Alexander horned sphere**. Although the solid horned sphere is not a manifold, R. H. Bing showed that its double (which is the 3-manifold obtained by gluing two copies of the horned sphere together along the corresponding points of their boundaries) is in fact the 3-sphere. One can consider other gluings of the solid horned sphere to a copy of itself, arising from different homeomorphisms of the boundary sphere to itself. This has also been shown to be the 3-sphere. The solid Alexander horned sphere is an example of a crumpled cube; i.e., a closed complementary domain of the embedding of a 2-sphere into the 3-sphere.

Generalizations

One can generalize Alexander's construction to generate other horned spheres by increasing the number of horns at each stage of Alexander's construction or considering the analogous construction in higher dimensions.

Other substantially different constructions exist for constructing such "wild" spheres. Another example, also found by Alexander, is Antoine's horned sphere, which is based on Antoine's necklace, a pathological embedding of the Cantor set into the 3-sphere.

See also

- Cantor tree surface
- Fox–Artin arc
- Platonic solid

Notes

1. Hocking & Young 1988, pp. 175–176. Spivak 1999, p. 55

References

- Alexander, J. W. (1924), "An Example of a Simply Connected Surface Bounding a Region which is not Simply Connected", *Proceedings of the National Academy of Sciences of the United States of America* (National Academy of Sciences) **10** (1): 8–10, doi:10.1073/pnas.10.1.8 (https://dx.doi.org/10.1073%2Fpnas.10.1.8), ISSN 0027-8424 (https://www.worldcat.org /issn/0027-8424), JSTOR 84202 (https://www.jstor.org/stable/84202)
- Bing, R. H. (1952), "A homeomorphism between the 3-sphere and the sum of two solid horned spheres", *Annals of Mathematics. Second Series* **56**: 354–362, ISSN 0003-486X (https://www.worldcat.org/issn/0003-486X), JSTOR 1969804 (https://www.jstor.org/stable /1969804), MR 0049549 (https://www.ams.org/mathscinet-getitem?mr=0049549)

- D. Fuchs, S. Tabachnikov, *Mathematical Omnibus: Thirty Lectures on Classic Mathematics*
- Hocking, John Gilbert; Young, Gail Sellers (1988) [1961]. *Topology*. Dover. ISBN 0-486-65676-4.
- Spivak, Michael (1999). *A comprehensive introduction to differential geometry (Volume 1)*. Publish or Perish. ISBN 0-914098-70-5.

External links

- Weisstein, Eric W., "Alexander's Horned Sphere" (http://mathworld.wolfram.com /AlexandersHornedSphere.html), *MathWorld*.
- Zbigniew Fiedorowicz. Math 655 – Introduction to Topology. [1] (http://www.math.ohio-state.edu /~fiedorow/math655/) – Lecture notes
- Construction of the Alexander sphere (http://www.youtube.com/watch?v=d1Vjsm9pQlc)
- rotating animation (http://www.ultrafractal.com/showcase/jos/alexanders-horn.html)
- PC OpenGL demo rendering and expanding the cusp (http://pouet.net/prod.php?which=30253)

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